Calculus II - Day 20

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Goals for today:

- Compute the work required to "build" a 3-D structure
- Compute the work required to pump fluid from a container

Work = force $\times$ distance
Important SI units:

- **Mass:** measured in kg
- **Distance:** measured in m
- **Force:** measured in N (Newtons)
  $1\,\text{N}$ = amount of force required to accelerate a $1\,\text{kg}$ mass by $1\,\text{m/s}^2$
- **Work:** measured in J (Joules)
  $1\,\text{J}$ = amount of work done by a $1\,\text{N}$ force acting over a distance of $1\,\text{m}$

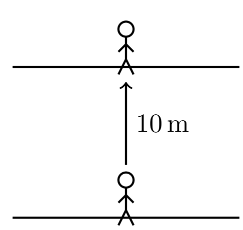
If F is constant: $W = F \cdot d$, but what if $F = F(x)$ changes as it's applied over a distance?
If $F(x)$ is the force applied at location x, then the total amount of work applied over the interval $[a, b]$ is:

$$W = \int_a^b F(x)\,dx$$

The amount of work required to lift an object of mass m a distance of y meters:

$$W = mgy \quad (\text{Force} = mg \cdot \text{distance} = y)$$

where $g = 9.8\,\text{m/s}^2$.
Example:

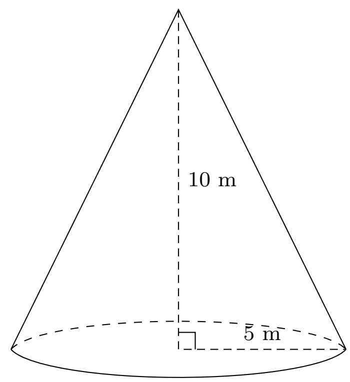


The amount of work done by an elevator to lift an 80kg man a distance of 10m is:

$$W = mgy = 80 \cdot g \cdot 10 = 800g\,\text{J}$$

Example:

Suppose you want to build a 3-D structure out of a material with density ρ (kg/m^3). To do this, different amounts of the material must be lifted different distances. How much total work is required?



Work required to lift this slice:

$$W = mgy$$

where:

$$m = V \cdot \rho = A(y) \cdot \Delta y \cdot \rho$$

$$A(y) = \text{cross-sectional area}, \quad \Delta y = \text{thickness of the slice}$$

Work for slice:

$$W = A(y)\Delta y \cdot \rho \cdot g \cdot y = \rho g A(y) y \Delta y$$

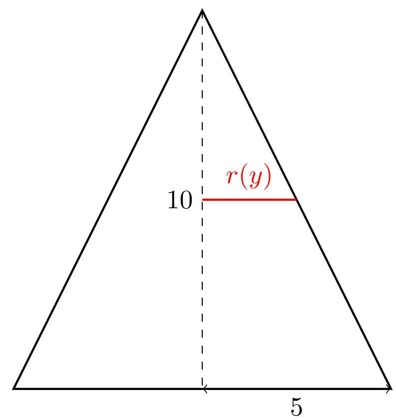
Dividing into n slices:

$$W = \sum_{k=1}^{n} (\rho g A(y_k) y_k \Delta y)$$

Taking the limit as $n \to \infty$:

$$W = \lim_{n\to\infty} \sum_{k=1}^{n} (\rho g A(y_k) y_k \Delta y) = \rho g \int_0^H A(y) y\, dy$$

What is the area $A(y)$?
Simpler Question: What is the radius?



$$r = 5 \quad \text{when } y = 0, \quad r = 0 \quad \text{when } y = 10$$

Radius decreases linearly:

$$\text{slope: } \frac{\Delta r}{\Delta y} = \frac{0-5}{10-0} = -\frac{1}{2}$$

Equation of the line:

$$r - r_0 = m(y - y_0) \quad \text{with } (y_0, r_0) = (0, 5)$$

$$r - 5 = -\frac{1}{2}(y - 0) \quad \Rightarrow \quad r = 5 - \frac{1}{2}y$$

Cross-sectional area:

$$A(y) = \pi r^2 = \pi(5 - \frac{1}{2}y)^2$$

$$W = \int_a^b \rho g A(y) y \, dy = \int_0^{10} \rho g \pi (5 - \frac{1}{2}y)^2 y \, dy$$

Expand $(5 - \frac{1}{2}y)^2$:

$$(5 - \frac{1}{2}y)^2 = 25 - 5y + \frac{1}{4}y^2$$

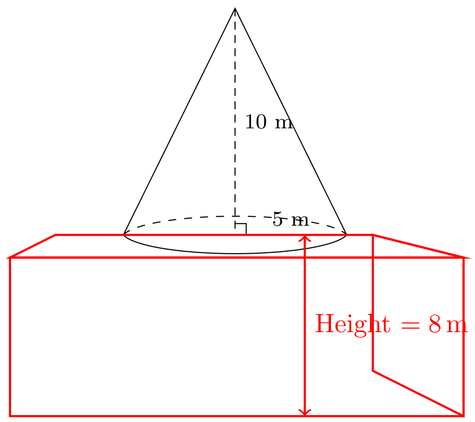
Substitute into the integral:

$$W = \rho g \pi \int_0^{10} \left(25y - 5y^2 + \frac{1}{4}y^3 \right) dy$$

$$W = \rho g \pi \left[\frac{25}{2}y^2 - \frac{5}{3}y^3 + \frac{1}{16}y^4 \right]_0^{10}$$

$$W = 6414.09\rho \, \text{J}$$

Example: What if we wanted to build this on an 8 m tall pedestal?



Previous Integral:

$$W = \int_a^b \rho g A(y) y \, dy$$

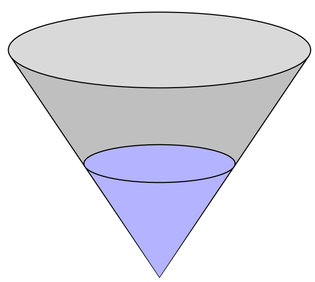
where $A(y) =$ area and $y =$ height lifted.

Here: Replace y for height with $y + 8$:

$$W = \int_0^{10} \pi \rho g (5 - \frac{1}{2}y)^2 (y+8) \, dy = 26,939.2 \, \rho \, \text{J}$$

More general formula: If the cross-section at height y has area $A(y)$ and must be lifted a distance $D(y)$, then:

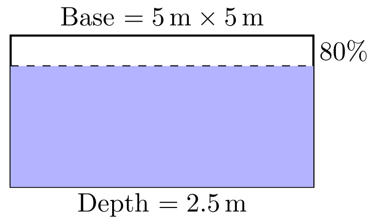
$$W = \int_a^b \rho g \, A(y) D(y) \, dy$$



Ex. Suppose you have a 2.5m deep swimming pool filled 80% of the way with water. How much work does it take to drain the pool from the top?

- **a)** The pool is a square prism with a 5 m × 5 m base.
- **b)** The pool is a cylinder with a radius of 3 m.

Note: cross-section is viewed horizontally.



a) Cross-sections are squares:

$$A(y) = 25 \quad \text{(constant)}$$

$$W = \int_a^b \rho g A(y) D(y)\, dy = \int_0^2 1000 \cdot 9.8 \cdot 25 \cdot (2.5 - y)\, dy$$

$$W = 1000 \cdot 9.8 \cdot 25 \int_0^2 (2.5 - y)\, dy$$

$$W \approx 18,375,000\,\text{J}$$

b) Cross-sections are disks:

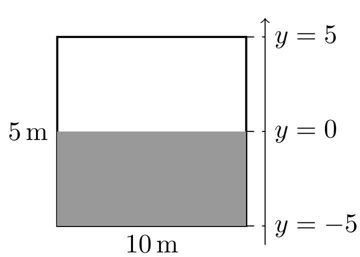
$$A(y) = 3^2\pi = 9\pi \quad \text{(constant)}$$

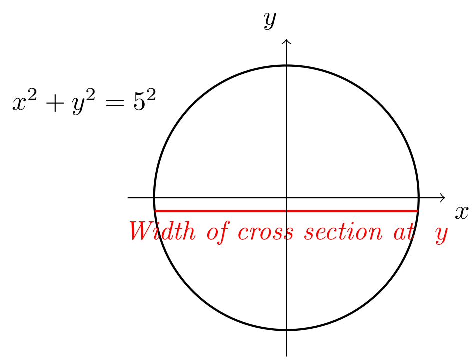
$$W = \int_0^2 1000 \cdot 9.8 \cdot 9\pi \cdot (2.5 - y)\, dy$$

$$W = 1000 \cdot 9.8 \cdot 9\pi \int_0^2 (2.5 - y)\, dy$$

$$W \approx 20,781,635\,\text{J}$$

Example: A cylindrical tank half-full of gasoline lies on its side.
If the cylinder has a length of 10 m, a radius of 5 m, and the density of gasoline is 737 kg/m^3, compute the work required to empty the tank through an outlet pipe on top of the tank.





$$\text{Circle formula: } x^2 + y^2 = 5^2 \text{...Want: Formula for } 2x$$

$$x = \sqrt{25 - y^2} \quad \Rightarrow \quad \text{Width} = 2x = 2\sqrt{25 - y^2}$$

$$A(y) = \text{Length} \times \text{Width} = 20\sqrt{25 - y^2}$$

$$W = \int_a^b \rho g A(y) D(y)\, dy = \int_{-5}^{0} 737 \cdot 9.8 \cdot 20\sqrt{25 - y^2} \cdot (5 - y)\, dy$$

$$\approx 20.2\,\text{million J}$$

$$W = \int_a^b \rho g A(y) D(y)\, dy$$

$$D(y) = \text{distance lifted}$$

$$\rho g A(y)\, dy = F = ma$$

$$\to m = \rho V = \rho A(y)\, dy$$

$$\to a = g$$